Quantum phase transitions and the hidden order in a two-chain extended boson Hubbard model at half-odd-integer fillings

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We study the phase diagram of two weakly coupled one-dimensional dipolar boson chains at half-odd-integer fillings. We find that the system contains a rich phase diagram. Four different phases are found. They are the Mott insulators, the single-particle resonant superfluid, the paired superfluid, and the bond-density or interchain density waves. Moreover, the Mott insulating phase can be further classified according to a hidden string order parameter, which is analogous to the one investigated recently in the one-dimensional boson Mott insulator at integer fillings.

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I. INTRODUCTION

The low dimensional electron and spin systems are usually regarded as important playgrounds for the studies of correlated quantum matters due to the strong quantum fluctuation effects in one and two spatial dimensions. Many exotic non-Fermi liquid states such as the Luttinger liquid and spin liquid states, to name just a few, are found in the single or coupled chain systems. However, nature never stops surprising us. The recent advances in the technique of loading ultracold gases into optical lattices open a new era in the research of condensed matter systems. Among the recent achievements, the experimental realization of the Boson-Hubbard model[1] signatures an important step in this direction, not only because of the tunability of the controlling parameters in the corresponding experiments but also because it facilitates the first observation of the Mott insulating state of bosons and the associated quantum phase transition.[2] While the atomic interactions in the ultracold gases can be treated as contact ones for most cases, a sizable longer range interaction is now within experimental reach by using the dipolar interaction among atoms,[3–5] which could provide further opportunities of controlling (designing) new experiments. Following these lines of developments, a natural question to ask is whether or not the longer range interactions can trigger a stable new quantum phase of matter which contains a nontrivial internal structure.

Among many research works devoted to the understanding of the effects of long-range dipolar interactions, we mention the recent work by Dalla Torre *et al.*,[6] who studied the one-dimensional boson insulators within the context of an extended boson Hubbard model (EBHM) by employing the density matrix renormalization group (DMRG) method. By tuning the ratio of the on-site interaction over the hopping amplitude U/t and the ratio of the longer range interaction over the hopping amplitude V/t, the mean field analysis shows that three different *conventional* phases can be reached. These include the Mott insulator at large U, the density wave state for large V, and a superfluid state for large t. The surprising thing is that a new intermediate insulating state, the Haldane insulator, which separates itself from the other two insulating states by second order quantum phase transitions was found in Ref. 6. Moreover, it was shown that such a state possesses a nonvanishing nonlocal string order, similar to the Haldane phase of the quantum spin-1 chain. Such a state is definitely beyond the reach of the traditional one-dimensional hydrodynamic effective theory for the one-dimensional boson superfluid-to-Mott transition.[7] Recently, the present author and his collaborators have developed a phenomenological *two-component* hydrodynamical effective theory,[8] which successfully captures the main features of all the phases, including the Haldane insulator, found in the recent DMRG study of the one-dimensional EBHM at integer fillings. This effective theory also clarifies the nature of the quantum phase transitions between different phases.

Knowing the above results, it is desirable to see if similar exotic insulating states can be found in other one-dimensional or quasi-one-dimensional systems. Coupled boson-chain systems have already been discussed in a number of previous publications,[9–11] and the related systems also attracted some renewed interests within the context of cold atoms in both one[12] and higher dimensions.[13] As far as we know, no exotic Mott insulating phase as we mentioned above has been noticed in these works. In this paper, we consider a dipolar boson system of two weakly coupled chains within the framework of the EBHM. Interestingly, we found that the competition of the interchain hopping and the interchain interaction does lead to two different types of Mott insulating states, with one of them possessing a nontrivial string order. In addition to that, we also found that the interchain attraction can give rise to an interesting *paired* superfluid state where the interchain bound boson pairs show an algebraic long-range superfluid order, while the single-boson superfluid correlations decay exponentially. Such a phase was also found in the previous discussions of coupled boson-chain systems under different conditions[9,10] and in the one-dimensional boson-boson and boson-fermion mixtures.[14–16] The rest of the paper is organized as follows. In Sec. II, we introduce our model and its effective theory. In Secs. III and IV, we analyze the phases of the model and discuss the issue of the string order in the Mott insulating state. Section V presents a strong interchain coupling analysis. The final section is dedicated to our conclusions, and the resulting phase diagram is summarized in Fig. 2.

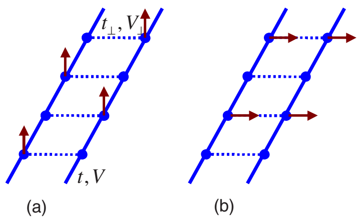

FIG. 1. (Color online) The two-chain lattice model described by the Hamiltonian [Eq. (1)]. The arrows denote the possible orientations of the dipole moments. (a) corresponds to the IDW phase and (b) represents the BDW phase considered in the paper (see Sec. III).

II. MODEL AND ITS EFFECTIVE THEORY

The lattice bosons studied in the present paper are described by the following extended Bose-Hubbard model:

$$H = - t \sum_{\sigma,\langle i,j\rangle} (b^\dagger_{\sigma i} b_{\sigma j} + \text{H.c.}) + \frac{U}{2}\sum_{\sigma i} \delta n^2_{\sigma i} + V \sum_{\sigma,\langle i,j\rangle} \delta n_{\sigma i}\delta n_{\sigma j} + t_\perp \sum_i (b^\dagger_{1i} b_{2i} + \text{H.c.}) + V_\perp \sum_i \delta n_{1i}\delta n_{2i}, \qquad (1)$$

where $\sigma=1,2$ is the chain index, t is the nearest-neighbor hopping along the chain, $t_\perp$ is the interchain hopping, U is the on-site repulsion, and $V, V_\perp$ are the nearest-neighbor interactions along and between the chains, respectively. The operator $b^\dagger_{\sigma i}$ creates a boson at site i on chain σ, $n_{\sigma i}=b^\dagger_{\sigma i}b_{\sigma i}$ is the number operator, and $\delta n_{\sigma i}=n_{\sigma i}-\bar{n}$ measures the deviation of the particle number from a mean filling $\bar{n}$. In the present paper, we will focus our attention to the cases of half-odd-integer fillings, i.e., $\bar{n}=N+1/2$, where N is a non-negative integer.

We stress that the above model can be realized by polar molecules[3,4] or atoms with a larger dipolar magnetic moment such as ^{53}Cr.[5] Moreover, it is possible to adjust the mutual orientations between the dipoles by using external electric (magnetic) fields, so that we may control the signs of the nearest-neighbor interactions V and $V_\perp$. Therefore, both the cases of the attractive and the repulsive nearest-neighbor couplings will be considered below (see Fig. 1). With this understanding in mind, we now discuss the low energy effective theory of the model defined in Eq. (1).

For simplicity, we first consider the case where the on-site repulsion U is the largest energy scale, i.e., $U \gg t, V, t_\perp, V_\perp$. Under this condition and near half-odd-integer fillings, we can truncate the boson Hilbert space so that only states with local occupations $n_{\sigma i}=N+1$ and N are allowed, and the boson operators can be mapped to spin-1/2 operators, $b^\dagger_{\sigma i} \to S^+_{\sigma i}$, $\delta n_{\sigma i} \to S^z_{\sigma i}$. In this low energy subspace, the extended boson Hubbard model is mapped to coupled xxz spin-1/2 models,

$$H = H_0 + H_\perp,$$

$$H_0 = - t\sum_{\langle i,j\rangle} (S^+_{1i}S^-_{1j} + \text{H.c.}) + V\sum_{\langle i,j\rangle} S^z_{1i}S^z_{1j} + (1 \to 2),$$

$$H_\perp = t_\perp \sum_i (S^+_{1i}S^-_{2i} + \text{H.c.}) + V_\perp \sum_i S^z_{1i}S^z_{2i}. \qquad (2)$$

In the following, we will assume that the interchain couplings $t_\perp, V_\perp$ are much smaller than the intrachain couplings t and V. Then, the spin-1/2 chain can be bosonized using the standard method,[17] and the intrachain Hamiltonian can be written in the following form:

$$H_0 = \sum_\sigma \frac{v}{2} \int dx \frac{1}{K}(\partial_x\phi_\sigma)^2 + K(\partial_x\theta_\sigma)^2 + g\int dx \cos\sqrt{4\pi}\phi_\sigma. \qquad (3)$$

At the perturbative level, $t \gg V$, we have $v=v_0\sqrt{1+\frac{2V}{\pi t}} \equiv \frac{v_0}{K}$, $v_0=ta_0$, and $g=\frac{V}{4\pi^2 a_0}$ (a_0 is the lattice spacing.). However, the validity of this effective action goes beyond its perturbative derivation, and the relations between the Luttinger parameter K and v and the spin-chain couplings t, V can be established exactly,[18]

$$v = v_0 \frac{\pi\sqrt{1-\Delta^2}}{2\cos^{-1}\Delta}, \quad K = \frac{\pi}{2(\pi - \cos^{-1}\Delta)}, \qquad (4)$$

with $\Delta=\frac{V}{2t}$ in the above equation.

The inclusion of the interchain coupling terms does not present any new difficulty. Upon including them, the resulting Hamiltonian becomes

$$H = \sum_{\alpha=s,a} \frac{u_\alpha}{2}\int dx \frac{1}{K_\alpha}(\partial\phi_\alpha)^2 + K_\alpha(\partial\theta_\alpha)^2 + 2g\int dx \cos\sqrt{8\pi}\phi_s \cos\sqrt{8\pi}\phi_a + \frac{t_\perp}{\pi a}\int dx \cos\sqrt{2\pi}\theta_a + \frac{V_\perp}{2\pi a_0}\int dx(\cos\sqrt{8\pi}\phi_a - \cos\sqrt{8\pi}\phi_s). \qquad (5)$$

In the above equation, ϕ_s, θ_s and ϕ_a, θ_a are the symmetric and antisymmetric combinations of the boson fields for the spin-1/2 operators, respectively. In terms of them, the original lattice boson operators can be expressed as

$$\frac{b_{1/2,i}}{\sqrt{a_0}} = \frac{1}{\sqrt{2\pi a_0}} e^{i\sqrt{\pi/2}(\theta_s \pm \theta_a)}[1 + (-1)^{x/a}\sin\sqrt{2\pi}(\phi_s \pm \phi_a)], \qquad (6)$$

and the bond and interchain density fluctuations are

$$\frac{\delta n_{s/a}}{a_0} = \sqrt{\frac{2}{\pi}}\partial_x\phi_{s/a} + (-1)^x\frac{-2}{\pi a_0}\begin{cases}\sin\sqrt{2\pi}\phi_s\cos\sqrt{2\pi}\phi_a \\ \cos\sqrt{2\pi}\phi_s\sin\sqrt{2\pi}\phi_a.\end{cases} \qquad (7)$$

Since the scaling dimension of the intrachain nonlinear term $g\cos\sqrt{8\pi}\phi_s\cos\sqrt{8\pi}\phi_a$, $2K_s+2K_a$, is equal to the sum of the scaling dimensions of the other two nonlinear terms due to $V_\perp$ in Eq. (5), it is always less relevant (in the renormalization group sense) than those interchain coupling terms in the weakly coupled chain limit. This indicates that we can safely neglect it in discussing the low energy properties of the system. Therefore, in the strong on-site repulsion limit, the two-chain extended boson Hubbard model at half-odd-integer fillings can be described by the following effective hydrodynamic theory:

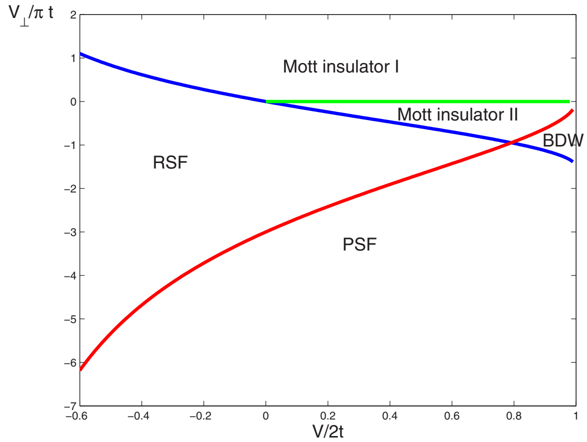

FIG. 2. (Color online) The phase diagram of the two weakly coupled half-filled one-dimensional EBHM obtained in the hard-core boson limit. Notice that for the hard-core bosons, only the BDW phase can occur, and this is because the two conditions of $(K_s<1, K_a<1/2)$ and $V_\perp>0$ cannot be satisfied simultaneously. However, we expect that this is an artifact of the hard-core condition, and for more general cases, depending on the sign of $V_\perp$, both the IDW and the BDW phases can be stabilized (see the discussions in Sec. V). Notice also that the phase transition across the red line is of the Ising universality class, while the phase transition across the blue line belongs to the KT universality class. Mott insulator II below the green line contains a nontrivial string order, and it is separated from the Mott insulating I state by a $U(1)$ Gaussian critical theory.

$$H=\sum_{\alpha=s,a}\frac{u_\alpha}{2}\int dx\frac{1}{K_\alpha}(\partial\phi_\alpha)^2+K_\alpha(\partial\theta_\alpha)^2+g_1\int dx\cos\sqrt{2\pi}\theta_a + g_2\int dx\cos\sqrt{8\pi}\phi_a+g_3\int dx\cos\sqrt{8\pi}\phi_s, \quad (8)$$

where $g_1=\frac{t_\perp}{\pi a_0}$, $g_2=-g_3=\frac{V_\perp}{2\pi a_0}$, $v_{s,a}=v\sqrt{1\pm K\frac{V_\perp a_0}{\pi v}}$, $K_{s,a}=K/\sqrt{1\pm K\frac{V_\perp}{\pi v}}$, and v, K here are parameters defined in Eq. (4). Notice that since the scaling dimensions of these nonlinear terms are determined by K_α, hence by V/t and $V_\perp/t$, the weak coupling phase diagram will depend on these two parameters, and not on $t_\perp$, as indicated in Fig. 2. At this point, it is interesting to notice that exactly an action of the same form occurs in our recent study of the one-dimensional boson Mott transition near integer fillings,[8] albeit in a completely different physical context.

Before we analyze the consequences of the Hamiltonian in Eq. (8), we emphasize that this effective action is quite general and is not restricted to hard-core bosons. For the finite U soft-core bosons, one can show that we will still get the same effective action [Eq. (8)] after some manipulations (see Appendix A.). Consequently, the results we get in this paper are robust as long as the interchain couplings are weak. The only problem is that in the soft-core boson case, the relations between the Luttinger parameters $K_{s,a}$ and the microscopic lattice couplings are, in general, not known and Eq. (8) can only be treated in a phenomenological manner.

III. PHASES AND PHASE TRANSITIONS OF THE EXTENDED BOSON HUBBARD MODEL

After establishing our effective theory for the two-chain EBHM, we now turn to study the possible phases and the associated quantum phase transitions of this system. Prior to that, we should notice that the microscopic symmetries of the original lattice model severely constrain the form of the low energy effective theory. For example, the global charge $U(1)$ symmetry, where the boson fields transform as $b_\sigma\to b_\sigma e^{i\varphi}$, forbids terms like $\cos\beta\theta_s$. There is also a discrete Z_2 symmetry which corresponds to the interchange of the two-chain indices $b_{1i}\leftrightarrow b_{2i}$, and this Z_2 symmetry forbids terms odd in ϕ_a or θ_a. These requirements are clearly obeyed by Eq. (8). In addition to that, there is a lattice translation symmetry of the EBHM. As we shall see immediately, this discrete translation symmetry can be broken spontaneously due to correlation effects. On the other hand, the continuous charge $U(1)$ symmetry can never be broken spontaneously in one dimension.

It is not hard to see from Eq. (8) that the only nonlinear term which can potentially open a gap in the symmetric mode is the g_3 term in Eq. (8), and its scaling dimension is $2K_s$. Therefore, we expect that depending on whether $K_s>1$ or $K_s<1$, the symmetric hydrodynamic mode will either be gapless or acquire a gap. Furthermore, since the ϕ_s and ϕ_a fields are decoupled at low energy, the gap opening transition of the symmetric mode from the $K_s<1$ to the $K_s>1$ region is completely characterized by a quantum sine-Gordon model of the ϕ_s field alone. Such a transition is known to be of the Kosterlitz-Thouless (KT) universality class.[19] On the other hand, for the antisymmetric mode, there are two competing nonlinear terms g_1 and g_2 in Eq. (8), which have scaling dimensions $\frac{1}{2K_a}$ and $2K_a$, respectively. Hence, depending on whether $K_a>1/2$ or $K_a<1/2$, either the $g_1\cos\sqrt{2\pi}\theta_a$ term or the $g_2\cos\sqrt{8\pi}\phi_a$ term will be dominant and opens a gap in the antisymmetric mode. In other words, the antisymmetric mode will always be gapped except at the critical line defined by $K_a=1/2$, where the critical theory falls into the Ising universality class. One way to see this is to "refermionize" the boson Hamiltonian.[17] However, the fact that the competition between a $\cos\sqrt{8\pi}\phi_a$ term and a $\cos\sqrt{2\pi}\theta_a$ term will result in an Ising transition can also be understood heuristically as follows: for $K_a<1/2$, we have $\langle\cos\sqrt{8\pi}\phi_a\rangle\neq 0$, while its "dual" $\langle\cos\sqrt{2\pi}\theta_a\rangle=0$. However, for $K_a>1/2$, the reverse is true. Exactly at the transition point $K_a=1/2$, the action of the antisymmetric modes is invariant under the duality transformation $\sqrt{2}\phi_a\to\frac{1}{\sqrt{2}}\theta_a$. The above behaviors resemble the well-known Krammer-Wannier duality of a two-dimensional classical Ising model or the one-dimensional quantum Ising chain in a transverse field. In particular, the duality symmetry of the effective action of the antisymmetric mode at the critical point $K_a=1/2$ is just the self-duality symmetry of the corresponding Ising models at their transition point. The above discussions suggest that, depending on $K_s\lessgtr 1$ and $K_a\lessgtr 1/2$, there can be four different phases. In the following, we shall elaborate on the natures of these four phases. Before we proceed, we remind the readers that tuning the parameters $K_{a,s}$ in the effective theory is related to tuning the parameters V/t and $V_\perp/t$ in the original lattice boson model (in the hard-core limit).

We start by considering the case where $K_s>1$ and $K_a>1/2$. In this case, the only relevant part of the interaction Hamiltonian is the g_1 term, which arises from the interchain Josephson coupling. Using Eqs. (6) and (7), one may easily see that the single-particle and density correlations are

$$\langle b_{1i}^\dagger b_{1j}\rangle = \langle b_{2i}^\dagger b_{2j}\rangle \sim \frac{1}{|i-j|^{1/4K_s}},$$

$$\langle \delta n_{\sigma i}\delta n_{\sigma j}\rangle \sim \frac{1}{|i-j|^2} + \text{const}\frac{(-1)^{|i-j|}e^{-|i-j|/\xi}}{|i-j|^{K_s}}, \qquad (9)$$

where ξ in the above equation is a finite correlation length of the gapped neutral mode. It is also interesting to examine the following long distance two-particle correlations:

$$\langle b_{1i}^\dagger b_{2i}^\dagger b_{1j} b_{2j}\rangle \sim \frac{1}{|i-j|^{1/K_s}}. \qquad (10)$$

From the above equations, we see that the single-particle correlation is dominant over the two-particle pair correlation, as is expected for a usual superfluid. On the other hand, from the boson representation $\langle b_{1i}^\dagger b_{2i}\rangle \sim e^{i\sqrt{2\pi}\theta_a}$, it is easy to see that the expectation value of $\langle b_{1i}^\dagger b_{2i}\rangle$ is nonzero due to the pining of the θ_a field by the g_1 term. The existence of a nonvanishing $\langle b_{1i}^\dagger b_{2i}\rangle$ in this one-dimensional system indicates that some kinds of resonant bonding bosons are formed for bosons on two different chains, and it is a direct manifestation of the phase locking between the two chains due to the interchain Josephson coupling g_1 term (see also the discussions in Sec. V). We also notice that although the density correlations along the chain decay algebraically, the density difference $\delta n_{-i}=\delta n_1-\delta n_2$ has an exponentially decaying correlation function. This is consistent with the above resonant bonding boson pair picture. Since there is no translational symmetry breaking in this phase, and motivated by Eq. (9) and the fact that $\langle b_{1i}^\dagger b_{2i}\rangle \neq 0$, this phase will be coined as the single-particle resonant superfluid (RSF) later in this paper.

We next consider the second gapless phase specified by the Luttinger parameters $K_s>1$ and $K_a<1/2$. In this case, the relevant part of the interacting Hamiltonian is the g_2 term which originates from the interchain dipolar interaction between bosons. The single-particle correlations decay exponentially, and density correlations in this phase behave like

$$\langle \delta n_{\sigma i}\delta n_{\sigma j}\rangle \sim \frac{1}{|i-j|^2} + (\text{const})\frac{(-1)^{|i-j|}}{|i-j|^{K_s}}. \qquad (11)$$

The most interesting feature of this phase is that although the symmetric mode is gapless and possesses a nonzero superfluid stiffness, the single-particle superfluid correlation decays exponentially. On the other hand, two-particle correlation shows an algebraic long-range order,

$$\langle b_{1i}^\dagger b_{2i}^\dagger b_{1j} b_{2j}\rangle \sim \frac{1}{|i-j|^{1/K_s}}. \qquad (12)$$

From the above results, we conclude that this translationally invariant phase is a one-dimensional *paired superfluid* (PSF), where the interchain bond-boson pairs flow coherently along the chain. As we have mentioned in the Introduction, this phase was also descovered in other binary boson systems, such as the coupled boson chains at incommensurate fillings[10] and in one-dimensional boson-boson mixtures.[14,15]

We now turn to discuss the remaining two gapped phases. For $K_s<1$ and $K_a>1/2$, both the interchain Josephson and the interchain dipolar interaction are relevant, and these interactions open gaps for both the symmetric and antisymmetric modes. The relevant part of the effective Hamiltonian in the present case is

$$H = \sum_{\alpha=s,a}\frac{v_\alpha}{2}\int dx\left[\frac{1}{K_\alpha}(\partial_x\phi_\alpha)^2 + K_s(\partial_x\theta_\alpha)^2\right] + g_1\int dx \cos\sqrt{2\pi}\theta_a + g_3\int dx\cos\sqrt{8\pi}\phi_s. \qquad (13)$$

In this phase, it is easy to see all the single-particle and density-density correlation functions decay exponentially. This is a translationally invariant insulating state with finite gaps to all its excitations. Therefore, it is a Mott insulator (MI).

The last phase corresponds to $K_s<1$, $K_a<1/2$, and the relevant part of the effective Hamiltonian in this case is

$$H = \sum_{\alpha=s,a}\frac{v_\alpha}{2}\int dx\left[\frac{1}{K_\alpha}(\partial_x\phi_\alpha)^2 + K_s(\partial_x\theta_\alpha)^2\right] + g_2\int dx \cos\sqrt{8\pi}\phi_a + g_3\int dx\cos\sqrt{8\pi}\phi_s, \qquad (14)$$

with $-g_3=g_2\propto V_\perp$ in the simplest EBHM. In this phase, all the single-particle and the pair correlations decay exponentially. On the other hand, for the bond-density and interchain density fluctuations, $\delta n_{\pm i}\equiv \delta n_{1i}\pm\delta n_{2i}$, we have

$$\begin{cases} V_\perp>0, & \langle \delta n_{-i}\rangle \sim (-1)^{x_i}/a_0 \\ V_\perp<0, & \langle \delta n_{+i}\rangle \sim (-1)^{x_i}/a_0. \end{cases} \qquad (15)$$

Notice that in the above results, a uniform background density was already subtracted due to the definition of $\delta n_{\sigma i}=n_{\sigma i}-\bar{n}$. This result indicates that the discrete lattice translation symmetry is broken spontaneously. Therefore, depending on the sign of the interchain dipolar interaction, the system exhibits either a bond-density wave (BDW) pattern or an interchain density wave (IDW) pattern (see Fig. 1 for a pictorial illustration). Notice that if we change the relative orientations of the dipoles between the two chains so that the interchain dipolar interaction changes its sign, we will tune a quantum phase transition across these two different density wave phases and the associated critical theory is a $U(1)\times U(1)$ Gaussian theory. These density wave states, unlike other states discussed in this paper, show long-range density wave orders. Therefore, we expect to see sharp peaks at the respective ordering wave vectors in their density-density correlation functions. It is interesting to see if we can detect this distinctive feature by measuring the noise-correlation spectrum of this system.[20]

IV. HIDDEN STRING ORDER

Inside the Mott insulator phase, we can also adjust the relative orientations of the dipoles between the two chains to change the sign of $V_\perp$. This will introduce a further quantum phase transition due to the accidental gap closing in the symmetric mode alone, and the corresponding critical theory is a $U(1)$ Gaussian theory. On both sides of the transition, the system is gapped and does not break any lattice symmetry. So, what is the difference between these two Mott insulating states? Interestingly, from the structure of the above hydrodynamic theory and the results of recent studies on the one-dimensional (1D) boson Mott insulator,[6,8] we expect that one of the Mott insulating phases will possess a nonvanishing nonlocal string order,[21] while the others will not. This can also be seen by noticing that our effective Hamiltonian bears some similarities with that of *isotropic* spin ladders,[22,23] where nontrivial string orders were also found. More specifically, drawing the analogy with the isotropic spin ladders, we may introduce the string operator

$$\hat{O}_{string}(i-j) = \lim_{|x-y|\to\infty} \langle \delta n_{+i} e^{i\pi\Sigma_{l=i+1}^{j-1}\delta n_{+l}} \delta n_{+j}\rangle. \tag{16}$$

Following the discussions of Nakamura,[24] we expect that the correct bosonized form of the string operator should be

$$\hat{O}_{string} \sim \lim_{|x-y|\to\infty} \langle \sin\sqrt{2\pi}\phi_s(x) \sin\sqrt{2\pi}\phi_s(y)\rangle, \tag{17}$$

From the structure of the nonlinear term in Eq. (13), we see that the term which is responsible for the gap opening in the symmetric mode is proportional to $-V_\perp \cos\sqrt{8\pi}\phi_s$. Hence, in the Mott phase and for $V_\perp<0$, the ϕ_s field will be pinned at $\sqrt{\pi/8}$, while for $V_\perp>0$, this field will be pinned at $\phi_s=0$. It is not hard to see that among the two Mott phases, only the one with $V_\perp<0$ possesses a nonvanishing string order of Eq. (17). In the language of coupled spin chains, the $V_\perp<0$ boson ladder model resembles a coupled spin chain with a ferromagnetic interchain Ising exchange coupling, while the $V_\perp>0$ boson model resembles a spin ladder model with an antiferromagnetic interchain Ising exchange coupling. Therefore, the boson Mott insulating state with a nontrivial string order that we found here is a boson analogy of the Haldane phase in a spin ladder system.[24]

A more interesting question is perhaps about how to detect the differences between the two Mott insulating states. From the structure of the effective theory, the bulk spectra of these two insulating states in the immediate neighborhood of the transition line are almost identical. On the other hand, from its analogy with the Haldane chain, we expect that the MI state with a nonvanishing string order contains a zero energy edge state.[25] Since all other bulk excitations have finite gapes, the existence of this zero energy edge state constitutes a unique feature of the string-ordered MI state. In real experiments, the system is trapped in a finite domain; it is interesting to see if we can detect it by measuring the Bragg spectroscopy,[26] as suggested in Ref. 6.

V. STRONG INTERCHAIN COUPLING ANALYSIS

Although the bulk part of this paper is devoted to the study of the weak interchain coupling limit, here, we briefly discuss the physical picture emerged from a strong interchain coupling analysis. Our purpose is to show that many features we found in previous sections can, in fact, be easily understood from a strong coupling point of view. This, among others, implies that the phases found in the weak interchain analysis can be extended to a larger region of the whole phase diagram in the parameter space.

We begin by considering the effect of a large $V_\perp$. The easiest way to study this is to consider the hard-core boson limit and keeping only the states with local occupations $n_{\sigma i}=0,1$. Under this condition, we may introduce the bond operators[27]

$$s^\dagger|0\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle), \quad t_x^\dagger|0\rangle = \frac{-1}{\sqrt{2}}(|11\rangle - |00\rangle),$$

$$t_y^\dagger|0\rangle = \frac{i}{\sqrt{2}}(|11\rangle + |00\rangle), \quad t_z^\dagger|0\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle), \tag{18}$$

for each site, and the ket, say, $|10\rangle$, denotes a state with a local bond-boson occupation $|n_{1i}=1, n_{2i}=0\rangle$, and these operators obey the constraint $s_i^\dagger s_i + \Sigma_\alpha t_{\alpha i}^\dagger t_{\alpha i}=1$. In terms of these bond operators, the interchain interaction can be written as

$$H_{V_\perp} = V_\perp \sum_i (t_{xi}^\dagger t_{xi} + t_{yi}^\dagger t_{yi}) - (s_i^\dagger s_i + t_{zi}^\dagger t_{zi}). \tag{19}$$

When $|V_\perp| \gg |t|, |t_\perp|$, and $V_\perp>0$, we may keep only s_i, t_{zi}, and the leading order effective low energy Hamiltonian becomes

$$H = -V_\perp + \frac{V}{2}\sum_i (n_{Ai}n_{Ai+1} + n_{Bi}n_{Bi+1}) - \frac{V}{2}\sum_i (n_{Ai}n_{Bi+1} + n_{Bi}n_{Ai+1}) + O\left(\frac{t^2, t_\perp^2}{V_\perp}\right), \tag{20}$$

where $A_i^\dagger = \frac{1}{\sqrt{2}}(s_i^\dagger + t_{zi}^\dagger), B_i^\dagger = \frac{1}{\sqrt{2}}(s_i^\dagger - t_{zi}^\dagger)$ create bond pairs $|10\rangle_i$ and $|01\rangle_i$, respectively, and $n_{A,Bi}=A_i^\dagger A_i, B_i^\dagger B_i$ are the number operators of the A and B bosons, respectively. We see from this strong coupling analysis that in the large $V_\perp$ limit, the ground state configuration is completely determined by minimizing the interparticle interactions between the A and B particles. Therefore, for $V>0$, we expect a Wigner crystal-like state, with the A and B particles arranged in an alternating pattern $\cdots ABAB \cdots$, while for $V<0$, phase separation will occur. In terms of the original two-chain boson language, this Wigner crystal-like density wave state is exactly the IDW phase discussed previously. When $V_\perp<0$, we get similar results, except now that the A and B operators are defined by $A_i^\dagger = \frac{1}{\sqrt{2}}(t_{xi}^\dagger - i t_{yi}^\dagger), B_i^\dagger = \frac{1}{\sqrt{2}}(t_{xi}^\dagger + i t_{yi}^\dagger)$, which create states with local bond occupations $|11\rangle_i$ and $|00\rangle_i$, respectively. Again, for $V>0$, the ground state is a density wave state and, for $V<0$, we have phase separations. It is not hard to see that the Wigner crystal-like state in this situation is just the BDW phase discussed earlier.

Similarly, when the interchain hopping amplitude $t_\perp$ is the largest energy scale, we may introduce the interchain bonding and antibonding boson fields $b_{\pm,i}=\frac{1}{\sqrt{2}}(b_{1i}\pm b_{2i})$. At low energy, the antibonding bosons can be projected out and the resulting low energy Hamiltonian becomes

$$H_{eff}=-t_\perp\sum_i b_{+i}^\dagger b_{+i}-t\sum_i (b_{+i}^\dagger b_{+,i+1}+\text{H.c.})+\frac{U}{4}\sum_i (n_{b_+i})^2 + \frac{V}{2}\sum_{\langle i,j\rangle} n_{b_+i}n_{b_+j}+O\left(\frac{U^2,V_\perp^2,V_\perp^2}{t_\perp}\right), \quad (21)$$

to the leading order, where $n_{b_+i}=b_{+i}^\dagger b_{+i}$ is the density operator of the bonding bosons. This effective Hamiltonian is nothing but that of the one-dimensional EBHM of the bonding bosons at integer fillings. Transforming back to the original two-chain boson language, it is not hard to see that the superfluid phase of the bonding boson at large t, where the boson density distributed uniformly, the operator $\langle b_{1i}^\dagger b_{2i} + \text{H.c.}\rangle$ possesses a nonzero expectation value. Hence, this phase may be identified as the RSF phase of the two-chain EBHM considered in this paper. Similarly, its density wave phase at large V can be viewed as the BDW phase of the original two-chain system. According to Ref. 6, when V increases to the order of U, we expect that the bonding boson will enter into a "Haldane insulator" phase which has a nontrivial string order. Since in the truncated Hilbert space where the antibonding bosons are projected out, the density operator n_{b_+i} coincides with the total bond-boson number operator n_{+i}. Hence, the string operator defined in Eq. (16) reduces to that of Ref. 6 in the large $t_\perp$ limit. Although the transition between the Haldane insulating phase and the conventional Mott insulating phase of the bonding bosons is achieved by tuning the intrachain coupling V, while it is the interchain coupling $V_\perp$ which plays a decisive role in the corresponding phase transition of the weakly coupled chain case, the fact that the string operators coincide in certain limit suggests that these two phases are, in fact, adiabatically connected.

VI. CONCLUSIONS AND DISCUSSIONS

We now briefly discuss some earlier results which are related to the present work. A phase analogous to our PSF state was noticed in the context of a coupled spin model,[9] and it was also shown to appear in a boson ladder model at incommensurate fillings.[10] Within the context of cold atoms, the coupled boson condensates with an interchain hopping term were studied within the framework of the quantum sine-Gordon model in Ref. 12. The charge mode in that system is always gapless and resembles the RSF phase in the present paper. The 1D binary mixtures of bosons or of a boson and a spin polarized fermion were studied in Ref. 15. It was found that for mixtures of two gases with equal densities, an attraction between the two components will enhance the pairing fluctuations. The attraction in Ref. 15 corresponds exactly to the g_2 term of our effective theory. The fact that the introduction of interspecies interaction in a 1D boson-boson mixture can induce the paired superfluid phase was also noticed in Ref. 14, and it was analyzed in terms of a two-component boson Hubbard model with on-site interaction only. It is also interesting to notice that a kind of composite fermion-boson pairing correlation was examined within the context of a 1D boson-fermion mixture,[15,16] and it was found that this pairing fluctuation was indeed enhanced due to the boson-fermion attraction. Since the one-dimensional fermions can be mapped to hard-core bosons through a Jordan-Wigner transformation, the effective action of the paired superfluid phase of the boson-fermion mixtures and that of the boson-boson mixtures are expected to be very similar.[15] In particular, in both cases, the paired superfluid phases are characterized by the same action that we discussed in Sec. III.

The Mott-superfluid transition for a boson ladder model, coupled by the interchain Josephson coupling alone, at the commensurate filling of one boson per site was studied in Ref. 11, while in our present case, the Mott insulating state occurs at half-odd-integer fillings. We also notice that the hidden order in single chain electronic Hubbard models was studied using both the bosonization and the DMRG methods in Ref. 28, and there are some recent efforts which address the issue of string orders in electronic band insulators and their relations with the Haldane chain.[29] These works, together with the results of Ref. 6 and our work, suggest that the idea of classifying the boson or fermion Mott insulators in quasi-one-dimensional systems using the string order may be quite general.

To summarize, the strong coupling analysis in Sec. V shows that the phases we obtained, when one of the interchain coupling is dominant, are consistent with the results we gained in the weak coupling bosonization analysis. Since one of the most important conclusion that follows from the bosonization study is that the interplay between the interchain hopping and interchain interaction results in two exotic phases—the paired superfluid state and the Mott insulating state with a string order—it is tempting to speculate that the major results of the bosonization analysis in this paper should be valid even when the interchain coupling strength is not so weak and can be extended to larger regions in the whole phase diagram.

Using the analytic results obtained in this paper as a guide, we hope that it will be helpful for future numerical works to determine the exact phase boundaries of the two-chain EBHM and, more importantly, the exotic insulating phase and the paired superfluid phase can be observed in the future experiments.

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APPENDIX A: DERIVATION OF THE EFFECTIVE THEORY

In this appendix, we briefly outline the derivation of our effective action [Eq. (8)] without imposing the hard-core boson condition right from the beginning.

It is well known[7] that a one-dimensional extended boson Hubbard model at half-odd-integer fillings can be described by the following effective theory:

$$H = \frac{v}{2}\int dx\left[\frac{1}{K}(\partial_x\phi)^2 + K(\partial_x\theta)^2 + g\cos 4\sqrt{\pi}\phi\right], \quad (A1)$$

where the phenomenological parameter v is the sound velocity, and the Luttinger parameters K are determined by the boson-boson interactions. The fields $\phi(x)$ and $\theta(x)$ are a pair of conjugate fields obeying the commutation relation $[\phi(x),\theta(y)]=i\theta(y-x)$. The nonlinear cosine term originates from the periodic lattice potential. For noninteracting bosons, the parameter K approaches infinity, while for hard-core bosons, K approaches 1, and the system is equivalent to a noninteracting spinless fermion model. For systems with longer range interactions, the Luttinger parameter K can be smaller than 1.[7] However, the exact relations between these parameters and a specific microscopic lattice boson model are generally unavailable. In the following, we shall treat them as phenomenological ones.

For two weakly coupled chains, we may introduce a pair of fields (ϕ_1,θ_1) and (ϕ_2,θ_2) for each chain and express the two-chain system in terms of the following phase action:

$$H = \sum_{\sigma=1,2}\frac{u_\sigma}{2}\int dx\left[\frac{1}{K}(\partial_x\phi_\sigma)^2 + K(\partial_x\theta_\sigma)^2 + g\cos 4\sqrt{\pi}\phi_\sigma\right]. \quad (A2)$$

Using the single-boson operator,[7]

$$\frac{b_\sigma}{\sqrt{a}} = \left(\rho_0 + \frac{1}{\sqrt{\pi}}\right)^{1/2}\sum_p e^{2ip(\pi\rho_\sigma x+\sqrt{\pi}\phi_\sigma)}e^{i\sqrt{\pi}\theta_\sigma}, \quad (A3)$$

and keeping only the most relevant terms (in the renormalization group sense), we found that, to leading order, the interchain hopping term becomes

$$\begin{aligned} t_\perp\sum_i (b_{1i}^\dagger b_{2i} + \text{H.c.}) &= \rho_0 t_\perp\int dx\left\{e^{i\sqrt{\pi}(\theta_1-\theta_2)}\left[\sum_p e^{2ip\sqrt{\pi}(\phi_1-\phi_2)}\right] + \text{H.c.}\right\} \\ &= 2\rho_0 t_\perp\int dx\cos\sqrt{\pi}(\theta_1-\theta_2) + \cdots. \end{aligned} \quad (A4)$$

Notice that in arriving at the second line in the above equation, we have dropped terms containing a spatially oscillatory phase factor since such terms are highly irrelevant under scaling. In arriving at the last line, we also dropped terms with higher harmonics, which are also less relevant than the leading term.

Similarly, using the following hydrodynamic representation of the boson density operator:

$$\frac{\delta n_{\sigma i}}{a} = \frac{n_{\sigma i}}{a} - \rho_0 = \frac{1}{\sqrt{\pi}}\partial_x\phi_\sigma + \rho_0\sum_p e^{2ip(\pi\rho_0 x+\sqrt{\pi}\phi_\sigma)}, \quad (A5)$$

and noticing that for the half-filling case $\rho_0=1/2$, the interchain interaction term becomes

$$\begin{aligned} V_\perp\sum_i \delta n_{1i}\delta n_{2i} &= \int dx\left\{\frac{V_\perp a}{\pi}[\partial\phi_1(x)][\partial_x\phi_2(x)] + V_\perp a\rho_0^2\sum_p e^{2ip\sqrt{\pi}(\phi_1-\phi_2)} + \text{H.c.}\right\} \\ &= \int dx\frac{V_\perp a}{\pi}(\partial_x\phi_1)(\partial_x\phi_2) \\ &\quad + 2V_\perp a\rho_0^2[\cos 2\sqrt{\pi}(\phi_1-\phi_2) \\ &\quad + \cos 2\sqrt{\pi}(\phi_1+\phi_2)] + \cdots. \end{aligned} \quad (A6)$$

Again, we have kept only the most relevant terms in the last line of the above equation. Putting these results together, we get

$$\begin{aligned} H &= \sum_{\sigma=1,2}\frac{u_\sigma}{2}\int dx\left[\frac{1}{K}(\partial_x\phi_\sigma)^2 + K(\partial_x\theta_\sigma)^2 + g\cos 4\sqrt{\pi}\phi_\sigma\right] \\ &\quad + 2\rho_0 t_\perp\int dx\cos\sqrt{\pi}(\theta_1-\theta_2) + \int dx\frac{V_\perp a}{\pi}(\partial_x\phi_1)(\partial_x\phi_2) \\ &\quad + \int dx 2V_\perp a\rho_0^2\cos 2\sqrt{\pi}(\phi_1+\phi_2). \end{aligned} \quad (A7)$$

In the symmetric-chain case studied in this paper, we may take $u_1=u_2=v$, $K_1=K_2=K$. Introducing the symmetric and antisymmetric combinations $\frac{1}{\sqrt{2}}(\phi_1\pm\phi_2)$ and absorbing the $\partial_x\phi_1\partial_x\phi_2$ term into the redefinitions of K, the above Hamiltonian can be written as

$$\begin{aligned} H &= \sum_{\alpha=s,a}\frac{u_\alpha}{2}\int dx\frac{1}{K_\alpha}(\partial\phi_\alpha)^2 + K_\alpha(\partial_x\theta_\alpha)^2 \\ &\quad + 2g\int dx\cos\sqrt{8\pi}\phi_s\cos\sqrt{8\pi}\phi_a \\ &\quad + 2\rho_0 t_\perp\int dx\cos\sqrt{2\pi}\theta_a + 2V_\perp a\rho_0^2 \\ &\quad \times\int dx(\cos\sqrt{8\pi}\phi_s + \cos\sqrt{8\pi}\phi_a), \end{aligned} \quad (A8)$$

where $u_{s,a}=v\sqrt{1\pm K\frac{V_\perp a}{\pi v}}$ and $K_{s,a}=K/\sqrt{1\pm KV_\perp a/\pi v}$. Since the scaling dimension of the $\cos\sqrt{8\pi}\phi_s\cos\sqrt{8\pi}\phi_a$ term, $2K_s+2K_a$, is equal to the sum of the scaling dimensions of the last two terms of the above equation, it is always less relevant (in the renormalization group sense) than the terms in the whole parameter region where the present weak coupling analysis is applicable. We can safely ignore this term, and the symmetric and antisymmetric modes are essentially decoupled in the low energy limit. The reader may notice that there is an apparent sign difference in the g_3 terms of Eq. (8) and the corresponding term in Eq. (A8). This difference originates from the difference of the bosonization convention we used for the spin operator and for the lattice boson operators. This will not lead to any difference in the conclusion of the paper. In particular, Eq. (15) was unaffected since the staggered parts of the density fluctuation operators $\delta n_{\pm j}$ becomes $\cos\sqrt{2\pi}\phi_s \cos\sqrt{2\pi}\phi_a$ and $\sin\sqrt{2\pi}\phi_s \sin\sqrt{2\pi}\phi_a$, respectively. Therefore, apart from some unimportant differences in the nonuniversal numerical coefficients, we recover Eq. (8).

APPENDIX B: CORRELATION FUNCTIONS

In this appendix, we briefly outline the derivations of the correlation functions that occurred in Sec. IV.

We will use Eq. (9) as an example to briefly discuss the calculations of the correlation functions in each phase of the Hamiltonian [Eq. (A8)]. In the phase where $K_s > 1$ and $K_a > 1/2$, the low energy property of the system is described by

$$H = \sum_{\alpha=s,a} \frac{u_\alpha}{2} \int dx \frac{1}{K_\alpha} (\partial\phi_\alpha)^2 + K_\alpha (\partial_x \theta_\alpha)^2 + g_1 \int dx \cos\sqrt{2\pi}\theta_a, \tag{B1}$$

i.e., the neutral mode is gapped and the field θ_a, depending on the sign of $t_\perp$, is pinned at 0 or $\sqrt{\pi/8}$. On the other hand, the symmetric mode remains gapless. The leading part of the single-particle correlation behaves as

$$\langle b_{1i}^\dagger b_{1j} \rangle \sim \langle e^{i\sqrt{\pi/2}(\theta_s+\theta_a)(x)} e^{-i\sqrt{\pi/2}(\theta_s+\theta_a)(0)} \rangle = e^{(\pi/2)\langle \theta_s(x)\theta_s(0)\rangle} \langle e^{i\sqrt{\pi/2}\theta_a(x)} e^{-i\sqrt{\pi/2}\theta_a(0)} \rangle. \tag{B2}$$

In arriving at the last line in the above equation, we have used two facts. First, the symmetric field and the antisymmetric field are decoupled in Eq. (B1). Second, the action of the symmetric field is a Gaussian one; hence, we have the identity $\langle e^{i\beta\theta_s(x)} e^{-i\beta\theta_s(y)} \rangle = e^{\beta^2 \langle \theta_s(x)\theta_s(y)\rangle}$, where β here is an arbitrary constant.[17] The second factor in the last line of the above equation behaves as a constant due to the pinning of the θ_a field, while the first factor gives a power law correlation, as indicated in Eq. (9), since $\langle \theta_s(x)\theta_s(0)\rangle \approx \frac{-1}{2\pi K_s} \ln|x|$ for $|x| \gg a$. The leading part of the density-density correlation behaves as follows:

$$\begin{aligned}\langle \delta n_{1i} \delta n_{1j} \rangle &= \frac{1}{\pi} \langle \partial_x \phi_1(x) \partial_x \phi_1(0) \rangle \\ &\quad + \text{const}(-1)^x \langle \sin\sqrt{4\pi}\phi_1(x) \sin\sqrt{4\pi}\phi_1(0) \rangle \\ &= \frac{1}{\pi} (\langle \partial_x \phi_s(x) \partial_x \phi_s(0) \rangle + \langle \partial_x \phi_a(x) \partial_x \phi_a(0) \rangle) \\ &\quad + \text{const}(-1)^x \langle e^{i\sqrt{2\pi}\phi_s(x)} e^{-i\sqrt{2\pi}\phi_s(0)} \rangle \\ &\quad \times \langle e^{i\sqrt{2\pi}\phi_a(x)} e^{-i\sqrt{2\pi}\phi_a(0)} \rangle. \end{aligned} \tag{B3}$$

The correlation $\langle \partial_x \phi_s(x) \partial_x \phi_s(0) \rangle$ decays algebraically, as indicated in Eq. (9), while the correlation function $\langle \partial_x \phi_a(x) \partial_x \phi_a(0) \rangle$ decays exponentially due to the pinning of its dual field θ_a. The staggered part also decays exponentially due to the same reason; hence, the system does not show density wave orders in this phase. The explicit evaluation of the correlation functions of the gapped neutral mode relies on the exact solution of the sine-Gordon model and such a complication is not necessary for our purpose since all we need to know is whether there is a (quasi)long-range order or not. On the other hand, to have a qualitative understanding, they may be estimated in a semiclassical approximation[7] by replacing the cosine term in the neutral mode with a mass term $\frac{1}{2} m_a \theta_a^2$, with $m_a \approx \pi |g_1|$. Within the semiclassical approximation, $\langle \theta_a(x)\theta_a(0)\rangle \sim e^{-m_a|x|}$ and $\lim_{|x|\to\infty} \langle e^{i\alpha\theta_a(x)} e^{-i\alpha\theta_a(0)} \rangle \to \text{const}$, while all other correlation functions involving the field ϕ_a decay like $e^{-|x|/\xi}$, with $\xi \approx m_a$ as the correlation length of the gapped neutral mode. The correlation functions in other phases can be analyzed in a similar manner.

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